Pure Appl. Geophys. 172 (2015), 605–606

DOI 10.1007/s00024-014-0975-8

Pure and Applied Geophysics

# Book Review

## Differential Geometry for Physicists and Mathematicians: Moving Frames and Differential Forms: From Euclid Past Riemann, José G. Vargas, World Scientific, 2014; ISBN 978-981-4566-39-1

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This book is for mathematical physicists and applied mathematicians and is concerned with the physical aspects of differential geometry written in the language and spirit of moving frames and differential forms. It consists of five parts and three appendices.

Part I (Introduction) presents the basic philosophy of the book.

In Chap. 1, the author summarizes the fundamental methods, ideas and rules of differential geometry, emphasizing that the principal aim of the book is various types of generalization attributable to E. Cartan.

Part II (Tools) is devoted to the presentation of the basic concepts of differential forms and differentiable manifolds as well as the elements of exterior calculus.

Chapter 2 concentrates on the differential forms. In particular, the controversy about the perception of this notion is clarified. The definition of the differential 1-form is presented here, with special focus on differential $r$-forms by means of the exterior product of a number $r$ of differential 1-forms. Next, the operation of the exterior product of 1-forms to $r$-forms together with the concept of invariance of differential forms is discussed. The metric aspects of differential forms (lengths, areas and volumes) are also briefly mentioned. Finally, two definitions of differentiable manifolds are formulated and elucidated.

Chapter 3 covers traditional material—vector spaces, dual tangent spaces, Euclidean vector spaces, tensor products and the elementary properties of Clifford algebras.

Chapter 4 deals with Cardan (or exterior) calculus. The visual notion of the exterior derivative (or exterior differential), based on the theorems of Gauss and Stokes, is presented first. Next, the formal definition of the exterior derivative, together with the effective expression for the exterior product of forms of any rank, is formulated. The coordinate independent definition of the exterior derivative is also introduced. After proving the Stokes theorem, the basic differential operators (gradient, curl, divergence and laplacian) are computed in terms of differential forms. The conservation law in the case of scalar-valued differential forms is exemplified in the context of electrodynamics. The notion of Lie algebra of a Lie group, which is illustrated with the example of the Maurer-Cartan form of the group, closes this part.

Part III (Two Klein Geometries) concentrates on affine and Euclidean geometries, respectively.

Chapter 5 contains the affine Klein geometry. The notion of affine connection is introduced first. Many basic concepts concerning frame bundles, fibers, connection differential forms, curvilinear coordinate systems, nonholonomic basis fields, etc., are discussed here, including Lie algebra of the affine group and the equations for continous groups (the Maurer-Cardan equations).

In Chap. 6, a terse survey of Euclidean space is given. The Euclidean frame bundles and associated bundles are defined together with several important examples. The meaning of covariance is investigated thoroughly as well as the Hodge star operator and Laplacian operator. The relationship between

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Euclidean structure and the integrability conditions together with the Lie algebra of a Euclidean group is also described. Finally, some remarks on Kähler calculus are presented.

Part IV (Cardan Connections) contains the connection theory of Cartan and its later development.

Chapter 7 is concerned with some examples of Cardan's generalization of Klein geometry. Two connections are discussed here, the Levi-Civita connection (LCC) and the teleparallel (or Columbus) connection. Various results concerning planes are included: the Euclidean 2-plane, the post-Klein 2-plane, the 2-sphere and the 2-torus. The so-called equivalence problem for Riemannian geometry is briefly sketched.

Chapter 8 deals with the affine connections. An affine connection is, roughly speaking, a connection on the tangent bundle of a (smooth) manifold. It presents the powerful differential-geometric machinery of Lie differentiation, the structure equation, curvature form, Bianchi identities, torsion interpretation, etc. An elegant and rigorous approach to affine connections is also proposed.

In Chap. 9, the rudiments of Euclidean geometry are presented. It begins with the discussion of manifolds with only metric structure. A definition of Euclidean connection is formulated together with the computation aspects of the LCC and the contorsion of Euclidean connections. Five rules for the calculation of the LCC by inspection are described and demonstrated. Some remarks on Finslerian torsion as well as the relation between Euclidean and Riemannian curvatures are also provided.

Chapter 10 discusses Cartan's contribution to geometry. The Cartan theory of connections unifies the Klein and Riemann programs of geometry. The basic philosophy of the Erlangen program of F. Klein is elucidated. Next, the so-called "false spaces of Riemann" are described, which are the manifolds with no positive definite metric. Cartan's treatment of the Riemannian curvature and his approach to solving the equivalence problem of Riemann are the themes of the next section. Then some basic facts about Riemannian space theory, together with its main physical applications, are reviewed. This information constitutes a neccessary background for comprehensive study of Einstein's differential 3-form. As an example, the classic result in general relativity, i.e., Schwarzschild's metric problem, is solved and analyzed.

Part V (The Future?) is dedicated to the historic aspects of Cartan ideas and their generalizations.

Chapter 11 discusses how basic concepts of classical geometry fit precisely into a universal geometric language of geometric algebra and calculus. The topics here include, among others, the Finslerian metric, Kaluza-Klein extension of geometry, Clifford algebra, Kähler calculus, etc.

Chapter 12 continues an exposition of the historical relations among algebra, calculus and geometry in the context of possible unification. The last chapter briefly describes the content of the planned new book by the author to familiarize the reader with its new thoughts.

In Appendix A, some facts are collected concerning curves and surfaces in 3D Euclidean space. Appendix B contains a short digression into the field of modern geometry from a historical point of view.

Short biographies of the two most influential geniuses who shaped the world of 20th century science, namely, E. Cartan (1869–1951) and H. Grassmann (1808–1877), are sketched. In Appendix C, a list of 51 publications, mainly by the author (J. Vargas) and D.G. Torr, is presented in reverse chronological order concerning Clifford algebra, Kähler calculus and Finsler bundles and closely related to the program of unification.

This is a very interesting, very deep book about a differential geometry. Before studying this book, the reader should have some prior familiarity with linear algebra, group theory, analysis, geometry and also modern theoretical physics. Still, if knowledge of this branch is helpful, even obligatory, to understand some parts of this book, another prerequisite seems to me even more essential, namely, an openness to and appreciation for the usually forgotten mathematical ideas of the past. Scientists completely unfamiliar with the concepts of Cartan may wish to consult Ivey and Landsberg (2003) (Cartan for beginners: differential geometry via moving frames and exterior differential systems. AMS, Providence, RI, 2003). I highly recommend the book.